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Note

Regular integral sum graphs

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Abstract

Given a set of integers S , $G(S) = (S, E)$ is a graph, where the edge uv exists if and only if $u + v \in S$. A graph $G = (V, E)$ is an integral sum graph or ISG if there exists a set $S \subset \mathbb{Z}$ such that $G = G(S)$. This set is called a *labeling* of G . The main results of this paper concern regular ISGs. It is proved that all 2-regular graphs with the exception of C_4 are integral sum graphs and that for every positive integer r there exists an r -regular ISG. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Integral sum graphs (ISG) were introduced by Harary [1]. A graph $G = (V, E)$ is an ISG if there exists a labeling $S(G) \subset \mathbb{Z}$ such that $V = S(G)$ and for every pair of distinct vertices $u, v \in V$, uv is an edge if and only if $u + v \in V$. The main subject investigated by Harary was the *integral sum number* $\zeta(G)$ —the minimal number of isolated vertices one must add to G to convert it into an integral sum graph. Authors would like to introduce another subject for investigation—the integral radius. For an ISG G , the *integral radius* $r(G)$ is the minimal natural number such that G can be labeled by $S(G) \subset \{-r(G), \dots, r(G)\}$. For a family of graphs F , let $r(F) = \max\{r(G) \mid G \in F \text{ and } G \text{ is ISG}\}$ whenever the maximum exists and $r(F) = \infty$ otherwise. Finally, for the family F_n of all graphs of order n we shall write $r(n)$ instead of $r(F_n)$. We think that a research of this parameter is important since it could allow us to apply computer

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methods for ISG investigations. For example, it would be enough to check all subsets of n integers from the interval $[-r(G), r(G)]$ (this can be easily done by a computer) in order to prove that a graph G on n vertices is not an ISG. The same argument works for the justification of the integral radius concept for families of the graphs. And especially useful will be finding a good upper bound for $r(n)$. Therefore, we want to ask

Question 1. *Is there a constant C , such that $r(n) \leq Cn$?*

So, throughout the whole paper the authors will try not only to prove that a given graph is an ISG, but also to find a labeling with the smallest possible integral radius.

Using the computer program mentioned above (checking all subsets of n integers lying in a given interval) authors were able to investigate the values of $r(n)$ for small n . We have the following:

$$r(1)=0, \quad r(2)=1, \quad r(3)=2, \quad r(4)=3, \quad r(5)=5, \quad r(6) \geq 7, \quad r(7) \geq 11, \quad r(8) \geq 13.$$

The justification of the equalities and inequalities above is following. All graphs on $n \leq 3$ vertices are ISG. There are 8 ISG on 4 vertices and 21 ISG on 5 vertices, and the straightforward verification proved that the rest 16 graphs on $n \leq 5$ vertices are not ISG. The authors have the catalogue of 70 ISG on 6 vertices, and among them 4 ISG have no labeling by $S \subset \{-6, \dots, 6\}$, and also the catalogue of 296 ISG on 7 vertices, and two of them have the integral radius equal to 11. Moreover, there are no other ISG on $n \leq 7$ vertices with the integral radius at most 15. For $n=8$, we know 4 ISG with the integral radius equal to 13.

Although we did not prove that $r(6)=7$, $r(7)=11$, and $r(8)=13$, the sequence 0, 1, 2, 3, 5, 7, 11, 13 above tempts one to ask

Question 2. *Is $r(n)$ equal to $(n-2)$ th prime number (for $n > 2$)?*

The authors have proved that Question 1 has positive answer for cycles, i.e. that $r(C_n) \leq Cn$ for some constant C .

In this paper, we shall study mostly regular ISG. Several results on this subject obtained earlier by the other researchers are listed below:

Statement 1 (Harary [1]). *All 1-regular graphs are integral sum graphs.*

Statement 2 (Xu Baogen [3]). *For every integer $m \geq 1$ the union of m 3-cycles mC_3 is an ISG.*

Statement 3 (Sharary [2]). *For every integer $k \geq 3$ except $k=4$ the cycle C_k is an ISG.*

Unfortunately, the labeling of cycles used by Sharary [2] yields an upper bound of the integral radius which is exponential with respect to n .

Section 2 contains several properties of ISG labelings, which will be of use in other sections. A labeling of all 2-regular graphs with the exception of C_4 is given in Section 3. This labeling yields an upper bound of the integral radius of the cycle C_n

which is linear with respect to n (but does not do it for an arbitrary 2-regular graph). The case of r -regular ISG for $r > 2$ is investigated in Section 4. It was proved, in particular, that for every such r an ISG exists.

2. Preliminaries

In this section, we observe several simple properties of ISG labelings. These properties will help us later.

Property 1. $S(G)$ contains zero if and only if G contains a vertex of degree $n - 1$.

Proof. It is evident that 0 must be adjacent to every vertex. Suppose that G contains a vertex of degree $n - 1$ whose label in $S(G)$ is equal to $v < 0$ (case $v > 0$ is similar). Denote by u the minimum label in $S' = S(G) \setminus \{v\}$. Then $u + v < u$, and therefore, $u + v \notin S'$. This means that $v + u = v$, and hence $u = 0$.

Property 2. If $G = (S, E)$ is an r -regular ISG of order n then $n = r + 1$ or $n \geq 2r + 1$.

Proof. If S contains zero, then $r = n - 1$ by the previous property. Otherwise, denote by $-a$ the maximal number in S by absolute value (we can suppose that it is negative). Then it has r distinct positive neighbors $b_1 < \dots < b_r$. Hence, S must contain at least $r + 1$ distinct negative numbers $-a < b_1 - a < \dots < b_r - a$. So, $n \geq 2r + 1$.

Property 3. C_4 is not an ISG.

It is a corollary of Property 2.

Property 4. Let $G = (S, E)$ be an ISG and $m \neq 0$ be an integer. Then $G(mS) = G$, where $mS = \{ms \mid s \in S\}$.

This property is evident.

Property 5. Let $G_1 = (S_1, E_1)$, $G_2 = (S_2, E_2)$ be two ISG and $0 \notin S_1 \cup S_2$. Then $G_1 + G_2$ is also ISG. ($G + H$ is just the union of two graphs G and H .)

Proof. Let $m_1 = \max\{|s|, s \in S_1\}$ and $m = 2m_1 + 1$. Then $G(S_1 \cup mS_2) = G_1 + G_2$.

3. Labelings of 2-regular graphs

In this section, we prove the following:

Theorem 1. Every 2-regular graph with the exception of C_4 is an integral sum graph. Moreover, there exists a labeling of C_n yielding the linear with respect to n integral radius.

Proof. By Property 3, C_4 is not an ISG. Due to Property 1, each labeling of C_3 (for example, $S = \{-1, 0, 1\}$) must contain zero. So, by Property 5, it is sufficient to prove that the following graphs are ISG (for every $k \geq 5$):

$$2C_3; 3C_3; 2C_4; 3C_4; C_3 + C_4; 2C_3 + C_4; 2C_4 + C_3; \quad (1)$$

$$C_k; C_k + C_3; C_k + C_4. \quad (2)$$

Note that some of these cases were proved by the other researchers earlier (see Statements 2 and 3 in the Introduction). Nevertheless, we present also our own labelings for these cases since they have a better integral radius, especially in the case of the cycles.

Next two lemmas complete the proof. Note that first two cases of Lemma 2 contain the labeling for cycles with a linear bound on the integral radius. However, using the construction from Property 5 destroys this linearity for an arbitrary 2-regular graph.

Lemma 1. *All graphs in (1) are integral sum graphs.*

Proof. We shall just indicate the labelings, leaving the verification to the reader.

$$S(2C_3) = \{-4, -3, -1, 1, 3, 4\},$$

$$S(3C_3) = \{-12, -7, -5, 1, 3, 4, 5, 7, 12\},$$

$$S(2C_4) = \{-5, -4, -2, -1, 2, 4, 5, 7\},$$

$$S(3C_4) = \{-28, -14, -6, 1, 5, 6, 9, 13, 14, 20, 22, 28\},$$

$$S(C_3 + C_4) = \{-5, -4, -2, -1, 1, 2, 4\},$$

$$S(2C_3 + C_4) = \{-10, -9, -7, -5, -3, 1, 5, 6, 8, 10\},$$

$$S(2C_4 + C_3) = \{-13, -12, -10, -9, -7, -4, 1, 6, 8, 10, 13\}.$$

All the labeling above are the best we know with respect to the integral radius (but maybe not the best possible for $3C_4$).

Lemma 1 is proved.

Lemma 2. *For every $k \geq 5$, all graphs in (2) are also integral sum graphs.*

Proof. Consider seven cases. All of them except the seventh have similar construction. Therefore, we give the detailed proof only for the first case, which is the most difficult case. For other cases we shall just indicate the labelings.

(1) For C_{2t+9} , $t \geq 1$, the labeling S includes the following numbers:

$$a_i = -17t - 1 + i, \quad i = 1, \dots, t + 2, \quad b_j = 4t - 1 + j, \quad j = 1, \dots, t + 1,$$

$$c = -t - 1, \quad d_1 = -12t, \quad d_2 = -12t + 1, \quad e = 17t, \quad f = 16t - 1, \quad g = -5t.$$

Note, that $a_1 < \dots < a_{t+2} < d_1 < d_2 < g < c < 0 < b_1 < \dots < b_{t+1} < f < e$.

Let us verify that this labeling is proper.

The number e is the largest, so there are no edges ef or eb_j ; $e + c = 16t - 1 = f$ —the first edge; $e + g = 12t \notin S$; $e + d_1 = 5t = b_{t+1}$ —the second edge; $e + d_2 = 5t + 1 \notin S$; finally, $0 \leq e + a_i \leq t + 1$ —no edges. So, there are two edges incident with e : ec and ed_1 .

There are no edges fb_j because $f + b_1 > e$; $f + c = 15t - 2 \notin S$; $f + g = 11t - 1 \notin S$; $f + d_1 = 4t - 1 \notin S$; there is an edge fd_2 , because $f + d_2 = 4t = b_1$; finally, $-t - 1 \leq f + a_i \leq 0$, hence, the only edge is fa_1 (because $f + a_1 = c$). So, f is adjacent only to d_2 and a_1 .

We have $3t - 1 \leq c + b_j \leq 4t - 1$ —no edges; $c + g = -6t - 1 \notin S$; $c + d_k = -13t - 2 + k \notin S$ for $k = 1, 2$; finally, $c + a_{t+2} = -17t = a_1$; and $c + a_i < a_1$ for $i < t + 2$.

There are no edges of type ga_i or gb_j because $-t \leq g + b_j \leq 0$; and $g + a_i < a_1$; but $g + d_k = a_k$, for $k = 1, 2$, so g is adjacent to both d_k .

There are no more edges incident with d_k , because $d_1 + d_2$ and $d_k + a_i$ are too small (less than a_1), and $-8t \leq d_k + b_j \leq -7t + 1$.

Finally, $a_i + a_j < a_1$; $8t + 1 \leq b_i + b_j \leq 10t - 1$ —no edges. But $a_i + b_j = -13t - 2 + i + j$; this sum lies between $-13t$ and $-11t + 1$. There are two numbers in S , namely d_1 and d_2 , inside this interval. So, $a_i b_j \in E$ if and only if $i + j \in \{t + 2, t + 3\}$. Therefore, we have the path $a_1 b_{t+1} a_2 b_t a_3 \cdots a_t b_2 a_{t+1} b_1 a_{t+2}$.

Adding the previously found edges, we obtain the following cycle:

$$fd_2gd_1eca_{t+2}b_1a_{t+1}b_2 \cdots b_1a_2b_{t+1}a_1.$$

(2) For C_{2t+8} , $t \geq 1$, the labeling S includes the following numbers:

$$a_i = t + 1 + i, i = 1, \dots, t + 1, \quad b_j = -5t - 9 + j, \quad j = 1, \dots, t + 2,$$

$$c_1 = -3t - 6, \quad c_2 = -3t - 5, \quad d_1 = -t - 2, \quad d_2 = -t - 1, \quad e = 4t + 7.$$

The cyclic order is following:

$$ec_2d_1c_1d_2b_{t+2}a_1b_{t+1}a_2 \cdots b_2a_{t+1}b_1.$$

(3) For $C_{2t} + C_3$, $t \geq 3$, the labeling S includes the following numbers:

$$a_i = 3t - 3 + i, \quad b_j = -2t + 1 + j, \quad i, j = 1, \dots, t,$$

$$c_1 = 2t - 2, \quad c_2 = 2t - 1, \quad d = -4t - 4.$$

Here, we have 3-cycle c_1c_2d and $2t$ -cycle

$$b_1a_1b_{t-1}a_2 \cdots a_{t-2}b_2a_{t-1}b_1a_t.$$

(4) For $C_{2t+1} + C_3$, $t \geq 3$, the labeling S includes the following numbers:

$$a_i = t - 2 + i, \quad i = 1, \dots, t + 1, \quad b_j = -4t + j, \quad j = 1, \dots, t,$$

$$c_1 = -2t, \quad c_2 = -2t + 1, \quad d = 4t - 2.$$

Here, we have 3-cycle c_1c_2d and $(2t+1)$ -cycle

$$b_1a_{t+1}b_2a_tb_3a_{t-1}\cdots a_3b_ta_2a_1.$$

(5) For $C_{2t} + C_4$, $t \geq 3$, the labeling S includes the following numbers:

$$a_i = t - 2 + i, \quad b_j = -5t + 4 + j, \quad i, j = 1, \dots, t,$$

$$c_1 = -3t + 3, \quad c_2 = -3t + 4, \quad d_1 = 4t - 3, \quad d_2 = 4t - 4.$$

The big cycle is

$$a_1b_ta_2b_{t-1}a_3\cdots b_3a_{t-1}b_2a_tb_1,$$

the 4-cycle is $c_1d_1c_2d_2$.

(6) For $C_{2t+1} + C_4$, $t \geq 2$, the labeling S includes the following numbers:

$$a_i = t + 1 + i, \quad i = 1, \dots, t + 1, \quad b_j = -3t - 5 + j, \quad j = 1, \dots, t,$$

$$c_1 = -t - 2, \quad c_2 = -t - 1, \quad d_1 = -2t - 2, \quad d_2 = 2t + 5.$$

We have a $(2t+1)$ -cycle

$$a_1a_2b_ta_3b_{t-1}\cdots a_tb_2a_{t+1}b_1,$$

and 4-cycle $c_1d_1c_2d_2$.

(7) This case includes the labelings for missed graphs, namely:

$$S(C_5) = \{-3, -2, -1, 1, 2\},$$

$$S(C_6) = \{-5, -4, -3, -1, 1, 4\},$$

$$S(C_7) = \{-6, -5, -4, -2, 1, 3, 7\},$$

$$S(C_8) = \{-7, -6, -4, -2, 1, 4, 5, 7\},$$

$$S(C_9) = \{-9, -8, -7, -4, -2, 1, 2, 4, 8\},$$

$$S(C_3 + C_5) = \{-7, -6, -4, -3, 2, 3, 4, 6\}.$$

Note that all these labelings are the best possible.

Lemma 2 is proved.

4. Regular ISG of higher degree

The natural question is: “Are there r -regular ISG for every $r \geq 3$?” The next theorem gives positive answer to this question.

Theorem 2. *For every positive integer $r \geq 0$ there exists an r -regular integral sum graph.*

Proof. Consider two cases:

1. $r = 2s + 1$. Let the labeling S contain the following sets of integers:

$$A = \{a_i = -2s + 1 + 2i, i = 0, 1, \dots, 4s + 1\},$$

$$B = \{b_j = -8s - 2 + 2j, j = 0, 1, \dots, 2s\},$$

$$C = \{c_k = -10s - 3 + 2k, k = 0, 1, \dots, 2s\}.$$

Let us prove that $G(S)$ is an r -regular graph. Note that A and C contain odd numbers and B contains even numbers. Using this property and the inequalities $a_0 + a_1 = -4s + 4 > b_{2s}$; $b_{2s} + b_{2s-1} = -8s - 6 < b_0$; $c_{2s} + c_{2s-1} = -12s - 8 < b_0$; $c_{2s} + b_{2s} = -10s - 5 < c_0$; we can observe that there are no edges either inside A, B , and C , or between B and C .

Let $t, j, k \in \{0, \dots, 2s\}$, $i \in \{0, \dots, 4s + 1\}$. There are three types of edges:

$$a_i + c_{2s-k} = -8s - 2 + 2(i - k) = b_{i-k}, \text{ for } 0 \leq i - k \leq 2s, \quad \text{type 1}$$

$$a_t + b_j = -10s - 1 + 2(t + j) = c_{t+j+1}, \text{ for } 0 \leq t + j \leq 2s - 1, \quad \text{type 2}$$

$$a_{4s+1-t} + b_j = -2s + 1 + 2(j - t) = a_{j-t}, \text{ for } t \leq j. \quad \text{type 3}$$

It is clear that there are no other edges between A and $B \cup C$. For a given k , we have exactly $2s + 1$ edges of type 1 incident with c_k ($i \in \{k, \dots, 2s + k\}$). For a given j , we have $2s - j$ edge(s) of type 2 ($t \in \{0, \dots, 2s - 1 - j\}$) and $j + 1$ edge(s) of type 3 ($t \in \{0, \dots, j\}$) incident with b_j . So, each vertex in $B \cup C$ has degree $2s + 1$. For $i = t$, there are $t + 1$ edge(s) of type 1 ($k \in \{0, \dots, t\}$) and $2s - t$ edge(s) of type 2 ($j \in \{0, \dots, 2s - 1 - t\}$); for $i = 4s + 1 - t$, we have t edges of type 1 ($k \in \{2s + 1 - t, \dots, 2s\}$) and $2s - t + 1$ edge(s) of type 3 ($j \in \{t, \dots, 2s\}$). In any case, a_i has degree $2s + 1$.

2. $r = 2s$. The labeling S consists of eight sets of integers:

$$A = \{a_i = 6s + i, i = 0, 1, \dots, 3s - 2\},$$

$$B = \{b_i = 20s + i, i = 0, 1, \dots, 3s - 2\},$$

$$C = \{c_j = 27s - 1 + j, j = 0, 1, \dots, 2s - 1\},$$

$$D = \{d_j = 30s - 2 + j, j = 0, 1, \dots, 2s - 1\},$$

$$E = \{e_k = -7s + 1 + k, k = 0, 1, \dots, s - 1\},$$

$$F = \{f_k = -21s + 1 + k, k = 0, 1, \dots, s - 1\},$$

$$G = \{g_k = -10s + 2 + k, k = 0, 1, \dots, s - 1\},$$

$$H = \{h_k = -24s + 2 + k, k = 0, 1, \dots, s - 1\}.$$

Note that $|A| = |B| = 3s - 1$, $|C| = |D| = 2s$, $|E| = |F| = |G| = |H| = s$.

Let us prove that $G(S)$ is an r -regular graph. If $j \in \{0, \dots, 2s - 1\}$, $k \in \{0, \dots, s - 1\}$ then $j + k \in \{0, \dots, 3s - 2\}$, and therefore all the following edges exist: $h_k + d_j = 6s + k +$

$j = a_{k+j}$, $g_k + d_j = 20s + k + j = b_{k+j}$, $f_k + c_j = 6s + k + j = a_{k+j}$, and $e_k + c_j = 20s + k + j = b_{k+j}$. On the other hand, straightforward verification shows that there are no more edges incident with vertices in C, D, E, F, G , or H . There are no edges inside A and B because $b_0 + b_1 = 40s + 1 > d_{2s-1}$ and $12s + 1 \leq a_i + a_j \leq 18s - 5$. But $a_i + b_j = 26s + i + j$, and so there are two types of edges:

$$i + j \in \{s - 1, \dots, 3s - 2\}, \quad \text{type } 1'$$

$$i + j \in \{4s - 2, \dots, 6s - 4\}. \quad \text{type } 2'$$

If $i < s$ then there are $2s$ edges of type $1'$ incident with a_i ($j \in \{s - 1 - i, \dots, 3s - 2 - i\}$) and no edges of type $2'$; otherwise, we have $3s - 1 - i$ edges of type $1'$ ($j \in \{0, \dots, 3s - 2 - i\}$) and $i - s + 1$ edges of type $2'$ ($j \in \{4s - 2 - i, \dots, 3s - 2\}$). In any case, a_i has degree $2s$. The same arguments work for b_j due to symmetry.

Theorem 2 is proved.

The next natural question about r -regular ISG is: “Is there any characterization of r -regular ISG for $r > 2$?” Even for $r = 3$, the condition of Property 2 is not sufficient to answer this question. This can be illustrated by the following theorem. Recall that the cube E^3 has 8 vertices and is 3-regular.

Theorem 3. *The cube E^3 is not an integral sum graph.*

Proof. Suppose that $E^3 = (S, E)$ is an ISG. Denote by $-a$ the maximal by absolute value number in S (we can suppose that it is negative), and let it be in the layer 0. Let layer k consists of all vertices which are at distance k from this vertex. It is clear that layers 1 and 2 contain three vertices each and layers 0 and 3 contain one vertex. So, the vertex $-a$ has three distinct positive neighbors b, c, d , which occupy the first layer. But then S must contain three negative numbers $(b - a)$, $(c - a)$, and $(d - a)$. Moreover, all of them must be in the layer 2. Indeed, if, for example, $(c - a)$ is not in the second layer, then $(b + c - a)$ and $(d + c - a)$ are not in S , and so, c is adjacent neither to $(b - a)$, nor to $(d - a)$, a contradiction. Similar argument works if $(b - a)$ or $(d - a)$ is not in the second layer.

Denote by x the vertex of the third layer. Let $b > c > d > 0$. Then $-a < d - a < c - a < b - a < 0$. The vertex d must be adjacent to at least one of the vertices $(b - a)$, $(c - a)$. But $b - a < b + d - a < d$, $c - a < c + d - a < d$, and $c + d - a \neq b - a$ (otherwise, $c + d = b$, a contradiction, because c and d are not adjacent). Hence, d must be adjacent to exactly one of these vertices and $x = b + d - a$ or $x = c + d - a$. In any case, S must contain the number $(b + c + d - 2a)$, because x is adjacent to the vertices $(b - a)$, $(c - a)$, and $(d - a)$. The set S must also contain the number $(c + b - a)$ (otherwise, $(c - a)$ and $(b - a)$ are not adjacent to b and c , respectively, and then both of them are adjacent to d , and we have already proved that it cannot be so). But $(c + b - a) + (d - a) = b + c + d - 2a$, and so the vertex $(c + b - a)$ is the neighbor of $(d - a)$. But $x < c + b - a < c$, and hence $c + b - a = d$ (the only available neighbor of $(d - a)$). But then $d - a = b + c - 2a = (b - a) + (c - a)$,—a contradiction, because $(b - a)$ and $(c - a)$ are not adjacent.

Theorem 3 is proved.

Question 3. *Is it true that N -dimensional cube E^N is not an ISG for every $N \geq 2$?*

Unfortunately, the straightforward technique of this proof is almost the only way to prove that given graph G is not an ISG. Therefore, the problem of characterization even for cubic ISG seems to be hard.

The authors have found the following cubic integral sum graphs. For $n = 8$, there is the only cubic ISG with

$$S = \{-6, -4, -2, -1, 2, 3, 4, 5\}.$$

For $n = 10$, we know 6 nonisomorphic cubic ISG:

$$S_1 = \{-9, -8, -7, -6, -4, -1, 1, 3, 5, 9\},$$

$$S_2 = \{-9, -8, -7, -5, -2, 1, 2, 6, 7, 9\},$$

$$S_3 = \{-10, -9, -6, -4, -3, -1, 1, 6, 7, 10\},$$

$$S_4 = \{-10, -7, -6, -5, -4, -1, 2, 4, 6, 9\},$$

$$S_5 = \{-11, -10, -8, -5, -3, -2, -1, 3, 8, 9\},$$

$$S_6 = \{-11, -9, -5, -4, -2, 1, 5, 6, 7, 9\}.$$

Finally, the authors have the catalogue of 41 nonisomorphic cubic ISG with $n = 12$.

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